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FUZZY ALGEBRA AND FUZZY VECTOR SPACE

S.SABIHA & M.Z.ALAM*
Research scholar, Dept. of Mathematics, L.N.M.U, (Darbhanga)

Abstract— In this paper, we have studied fuzzy algebra and fuzzy vector space. We examine the property of fuzzy algebra as defined by KANDEL,A and BAYATT,W.J [1]. The notion of fuzzy vector space was introduced by KATSARAS,A.K and LIU,D.B [2]. Using the definition of fuzzy vector space, we established a few results, when an element is added to fuzzy vector space and continuing this we define fuzzy subspace and also established the facts that the sum of two subspaces is also a subspace; this is also true in the case of scalar multiplication.

Keywords— Fuzzy algebra, Fuzzy vector space, Fuzzy subspace.

1. INTRODUCTION :

The concepts of fuzzy set was introduced by Zadeh [6],and the notion of fuzzy vector space was defined and established by KATSARAS,A.K and LIU,D.B [2]. Using the definition of fuzzy vector space and fuzzy subspace as defined by KATSARAS,A.K and LIU,D.B. We have established some results in the theorem 3.4 and theorem 4.1, respectively.

2. BASIC DEFINITION AND PROPERTIES

FUZZY ALGEBRA

A fuzzy algebra is defined to be the system Z = (Z,+,*,-), where Z has at least two distinct elements and for all x, y, z $\in Z$, system Z satisfies the following set of axioms.

(i) Idempotency : $x + x = x,\ \ x * x = x$
(ii) Commutativity : $x + y = y + x,\ \ x * y = y * x$
(iii) Associativity: $(x + y) + z = x + (y + z),\ \ (x * y) * z = x * (y * z)$
(iv) Absorption : $x + (x * y) = x,\ \ x * (x + y) = x$
(v) Distributivity : $x + (y * z) = (x + y) * (x + z),\ \ x * (y + z) = (x * y) + (x * z)$
(vi) Complement : If $x \in Z$, then there is a unique complement $\bar{x}$ of x, such that $\bar{x} \in Z$, and $\bar{\bar{x}} \in x$
(vii) Identities : $\exists\, e_+$,such that $x + e_+ = e_+ + x = x$, for all x

Also, there exist e_*, such that $e_* * x = x * e_* = x$, for all x

(vIII) De- Morgan law : $\overline{x + y} = \bar{x} * \bar{y},\ \ \overline{x * y} = \bar{x} + \bar{y}$

This system forms a distributive lattice with existence of unique identities under the operation '+' and '$*$' . But it is not Boolean algebra because the law of Boolean algebra

$x + \bar{x} = 1$, for all x and $\exists\, x$, such that $x * \bar{x} = 0$, are not true in fuzzy algebra. Hence every Boolean algebra is a fuzzy algebra but not vice-versa.

DEPT.OF MATHEMATICS,MILLAT COLLEGE,DARBHANGA-846004

Obviously, the class of fuzzy sets on any universe forms a fuzzy algebra with respect to union, intersection, and complement as defined by Zadeh,L.A [6]. The identity elements being the fuzzy null set ∅ and the whole set X.

Another fuzzy algebra is defined by the system Z = ([0,1],+, ∗,-), where +, ∗ and '- ' are interpreted as max, min, and complement, ($\bar{x} = 1 - x$, $\forall x \in [0,1]$), respectively.

3. FUZZY VECTOR SPACE

Let X be a vector space over K, where K is the space of real or complex numbers, then the vector space X equipped with addition '+' and scalar multiplication (.) defined over the fuzzy sets on X as below is called the fuzzy vector space.

Addition (+) : Let $A_1,\ldots\ldots.,A_n$ be fuzzy sets on vector space X, let $f : X^n \to X$, such that

$$f(x_1,\ldots\ldots,x_n) = x_1 + \ldots\ldots + x_n \text{, we define}$$

$$A_1 + \ldots\ldots + A_n = f(x_1,\ldots\ldots,x_n) \text{,by extension principle}$$

$$\mu_{f(A_1,\ldots\ldots,A_n)}(y) = \sup_{\substack{x_1,\ldots\ldots,x_n \\ y=f(x_1,\ldots\ldots,x_n)}} \min\{\mu_{A_1}(x_1),\ldots\ldots,\mu_{A_n}(x_n)\}$$

Obviously, when sets $A_1,\ldots\ldots A_n$, are ordinary sets, gradation functions used in the sum are taken as characteristic function of the set.

Scalar multiplication (.) : If α is a scalar and B be a fuzzy set in X and g : X → X, such that g(x) = αx, then using extension principle we define α B = g(B)

Where, $\mu_{g(B)}(y) = \sup\limits_{\substack{x \\ y=g(x)=\alpha x}} \{\mu_B(x)\}$, if y = α x

$\mu_{g(B)}(y) = 0$, if $y \neq \alpha x$ for any x

$\mu_{\alpha B}(y) = \sup\limits_{\substack{x \\ y=\alpha x}} \mu_B(x)$, y ϵ X

$\mu_{\alpha B}(y) = 0$, if y ≠ α x, for any x

THEOREM 3.1 : If E is a vector space over a field K, g is a mapping from E to E such that g(x) = α x, and B be a fuzzy set in X, then

(a) For any scalar α ≠ 0, $\mu_{\alpha B}(y) = \mu_B\left(\frac{1}{\alpha} y\right)$, for all $y \in E$

And for α =0, $\mu_{\alpha B}(y) = 0$, if y ≠ 0

$\mu_{\alpha B}(y) = \sup\limits_{x} \mu_B(x)$, if y = 0

(b) For all scalar α and for all $y \in E$, $\mu_{\alpha B}(y) \geq \mu_B(y)$

Proof : (a) Let α ≠ 0, by definition, $\mu_{\alpha B}(y) = \sup_{\substack{x \\ y=g(x)=\alpha x}} \{\mu_B(x)\} = \mu_B\left(\frac{1}{\alpha}y\right)$, for all y ∈ E

If α = 0, and y ≠ 0, then $g^{-1}(y) = \phi$,therefore $\mu_{\alpha B}(y) = 0$, by extension principle

If α = 0, and y =0, then $g^{-1}(y) \neq \phi$, as 0 = 0 x for all x

$\therefore \mu_{\alpha B}(y) = \sup_{\substack{x \\ y=0x}} \{\mu_B(x)\}$, holds for all x

$\therefore \mu_{\alpha B}(y) = \sup_{x} \mu_B(x)$, if (y = 0) and α =0

(b) For all scalar α, $\mu_{\alpha B}(y) = \sup_{\substack{x \\ y=\alpha x}} \{\mu_B(x)\}$, x ∈ E, y ∈ E

$\therefore \mu_{\alpha B}(\alpha x) = \sup_{\substack{x \\ \alpha x=\alpha x}} \{\mu_B(x)\}$, x ∈ E

$\therefore \mu_{\alpha B}(\alpha x) \geq \mu_B(\alpha x)$, for all α x ∈ E

i.e $\mu_{\alpha B}(y) \geq \mu_B(y)$, for all y ∈ E, [take α x = y]

THEOREM 3.2 : If E and F are vector spaces over K, f is a linear mapping from E to F and A, B are fuzzy sets on E

(i) $f(A+B) = f(A) + f(B)$

(ii) $f(\alpha A) = \alpha f(A)$, for all scalars α

i.e $f(\alpha A + \beta B) = \alpha f(A) + \beta f(B)$, where α, β are scalars

Proof (i) : Let $M = \{f(x) : x \in E\}$, let $a = \mu_{f(A+B)}(y)$,and $b = \mu_{f(A)+f(B)}(y)$

1st.Case : Let $y \notin M$, then obviously from the extension principle and the above definition of sum of fuzzy sets

$a = \mu_{f(A+B)}(y) = 0$, since $f^{-1}(y) = \phi$, let $y_1 + y_2 = y$, where $y_1, y_2 \in F$, then at least one of y_1, y_2, is not in M, otherwise $y_1 + y_2 \in M$

$\therefore \mu_{f(A)}(y_1) = 0$, or $\mu_{f(B)}(y_2) = 0$

Now $b = \mu_{f(A)+f(B)}(y) = \sup_{\substack{y_1, y_2 \\ y=y_1+y_2}} \min\{\mu_{f(A)}(y_1), \mu_{f(B)}(y_2)\} = 0$

$\therefore a=b$ i.e $\mu_{f(A+B)}(y)=\mu_{f(A)+f(B)}(y)$.

2nd Case : If $y\in M, a=\mu_{f(A+B)}(y)=\sup\limits_{\substack{x\\ y=f(x)}}\mu_{A+B}(x),\ x\in E$

$\therefore$ for $\varepsilon>0$, there exists x ∈ E, such that $\mu_{A+B}(x)>a-\varepsilon$, and $f(x)=y$

i.e $\min\{\mu_A(x_1),\mu_B(x_2)\}>a-\varepsilon$, for some x_1,x_2 with $x_1+x_2=x$, and $f(x_1+x_2)=f(x)=y$

i.e $f(x_1)+f(x_2)=y$, Now, $b=\mu_{f(A)+f(B)}(y)=\sup\limits_{\substack{y_1,y_2\\ y=y_1+y_2}}\min\{\mu_{f(A)}(y_1),\mu_{f(B)}(y_2)\}$

$\therefore b\geq\min\{\mu_{f(A)}(f(x_1)),\mu_{f(B)}(f(x_2))\}$, as $f(x_1)+f(x_2)=y$

i.e $b\geq\min\{\mu_A(x_1),\mu_B(x_2)\}>a-\varepsilon, \because \mu_{f(A)}(f(x_1))=\sup\limits_{f(x_1)=f(z)}\mu_A(z)$

$\therefore \mu_{f(A)}(f(x_1))\geq\mu_A(x_1)$, But ε is arbitrary, therefore we get $b\geq a$

Again given $\varepsilon>0$,there exists $y_1,y_2\in F$, such that $y_1+y_2=y$

And $\min\{\mu_{f(A)}(y_1),\mu_{f(B)}(y_2)\}>b-\varepsilon$, [we take $\varepsilon<b$]

i.e $\min\left\{\sup\limits_{\substack{x_1\\ y_1=f(x_1)}}\mu_A(x_1),\ \sup\limits_{\substack{x_2\\ y_2=f(x_2)}}\mu_B(x_2)\right\}>b-\varepsilon$, there exists $x_1,x_2\in E$, such that $f(x_1)=y_1$,and

$f(x_2)=y_2$, and $\min\{\mu_A(x_1),\mu_B(x_2)\}>b-\varepsilon$

$\therefore a>b-\varepsilon$, but ε, is arbitrary, therefore $a\geq b$, hence a = b

Proof (ii) : Let $M=\{f(x):x\in E\}$, let $c=\mu_{\alpha f(A)}(y)$,and $d=\mu_{f(\alpha A)}(y)$

If $y\notin M$, then $f^{-1}(y)=\phi$,and $\mu_{f(\alpha A)}(y)=\sup\limits_{\substack{x\\ y=f(x)}}\mu_{\alpha A}(x)=0$, if $f^{-1}(y)=\phi$

$\therefore d=0$, Now $c=\mu_{\alpha f(A)}(y),\ y\in F$

$\therefore c=\mu_{f(A)}\left(\dfrac{y}{\alpha}\right)$, ($\alpha\neq 0$)

$c = \sup\limits_{\substack{x \\ y/\alpha = f(x)}} \mu_A(x)$, but $f^{-1}(y) = \phi$, therefore c = 0, [as $f^{-1}(y/\alpha) = \phi$], where $\alpha \neq 0$

Hence , $c = d = 0$

Again suppose $y \in M$,and $\alpha \neq 0$

$$\therefore c = \mu_{\alpha f(A)}(y) = \mu_{f(A)}\left(\frac{1}{\alpha}\right)(y) = \sup_{\left(\frac{1}{\alpha}\right)y = f(x)} \mu_A(x)$$

$\Rightarrow c = \sup\limits_{y = f(\alpha x)} \mu_{\alpha A}(\alpha x)$, i.e $c = \sup\limits_{\substack{z \\ y = f(z)}} \mu_{\alpha A}(z) = d$

If $\alpha = 0$, and $y \neq 0$, $c = \mu_{\alpha f(A)}(y) = 0$, and $d = \sup\limits_{\substack{x \\ y = f(x)}} \mu_{\alpha A}(x) = 0$, [as $x \neq 0$,when $y \neq 0$]

$\therefore c = d$

If α = 0 and y = 0, $\therefore c = \mu_{\alpha f(A)}(y) = \sup\limits_{z} \mu_{f(A)}(z), z \in F$

$\Rightarrow c = \sup\limits_{x \in E} \mu_A(x), x \in E$, and $d = \sup\limits_{0 = f(x)} \mu_{\alpha A}(x)$, here y = 0

$\Rightarrow d = \mu_{\alpha A}(0) = \sup\limits_{x \in E} \mu_A(x), therefore, c = d$

THEOREM 3.3 : If A and B are fuzz sets in a vector space E over a field K, then for all scalars α

$$\alpha(A+B) = \alpha A + \alpha B$$

Proof : Let $f : E \to E$, such that, $f(x) = \alpha x$,obviously f is linear mapping from E to E, let A,B be any fuzzy sets in E, then

$\alpha A = f(A)$, and $\alpha B = f(B)$, by the definition of scalar product

$\therefore \alpha A + \alpha B = f(A) + f(B) = f(A+B) = \alpha(A+B)$,Since f is linear mapping ,and A + B , is fuzzy set in E.

DEFINITION 3.3 : If A is a fuzzy set in a vector space E, and x ∈ X we define x + A, as $x + A = \{x\} + A$

THEOREM 3.4 : If $f_x : E \to E$,(vector space) such that $f_x(y) = x + y$,then B is a fuzzy set in E and A is an ordinary subset of E, the following hold

(i) $x+B=f_x(B)$

(ii) $\mu_{x+B}(z)=\mu_B(z-x)$

(iii) $A+B=\bigcup_{x\in A}(x+B)$

PROOF : (I) We have , $\mu_{\{x\}+B}(z)=\sup_{\substack{x_1,x_2\\ z=x_1+x_2}}\min\{\mu_{\{x\}}(x_1),\mu_B(x_2)\}$

$$\mu_{\{x\}+B}(z)=\sup_{z=x+x_2}\min\{1,\mu_B(x_2)\}$$

$$\mu_{\{x\}+B}(z)=\sup_{\substack{x_2\\ z=f_x(x_2)}}\mu_B(x_2)$$

$$\mu_{\{x\}+B}(z)=\mu_{f_x(B)}(z)$$

Since, $f_x:E\to E$,and B is a fuzzy set in E, therefore $x+B=f_x(B)$

(ii) $\mu_{x+B}(z)=\mu_{f_x(B)}(z)=\sup_{z=f_x(y)}\mu_B(y)=\sup_{z=x+y}\mu_B(y)$, $[\because f_x(y)=x+y]$

$\therefore \mu_{x+B}(z)=\mu_B(z-x)$

(iii) $\mu_{A+B}(z)=\sup_{\substack{x_1,x_2\\ z=x_1+x_2}}\min\{\mu_A(x_1),\mu_B(x_2)\}$

$$\mu_{A+B}(z)=\sup_{\substack{x_1,x_2\\ z=x_1+x_2}}\min\mu_B(x_2)$$

$$\mu_{A+B}(z)=\sup_{x_1\in A}\mu_B(z-x_1)=\sup_{x\in A}\mu_{x+B}(z)$$

$$\therefore A+B=\bigcup_{x\in A}(x+B)$$

4.FUZZY SUBSPACE

DEFINITION 4.1 : A fuzzy set F in a vector space E is called fuzzy subspace of E if

(i) $F+F\subset F$

(ii) $\alpha F\subset F$, for every scalar α

THEOREM 4.1 : If A, B are fuzzy subspaces of E and K is a scalar. Then A + B and K A are fuzzy subspaces.

Proof : Since A and B are fuzzy subspaces of E

$A+A\subset A$, and $B+B\subset B$, now for all x_1,x_2,x_3,x_4 in E, we have

$\mu_A(x_1+x_2)\geq\min\{\mu_A(x_1),\mu_A(x_2)\}$ ....(i)

$\mu_B(x_3+x_4)\geq \min\{\mu_B(x_3),\mu_B(x_4)\}$ ....(ii)

Now we have , $\mu_{A+B}(z)=\sup_{z=x_1+x_3}\min\{\mu_A(x_1),\mu_B(x_3)\}$

i.e, $\mu_{A+B}(x_1+x_3)\geq \min\{\mu_A(x_1),\mu_B(x_3)\}$

and $\mu_{A+B}(x_2+x_4)\geq \min\{\mu_A(x_2),\mu_B(x_4)\}$

We have to show that , $A+B+A+B\subset A+B$

$$\mu_{A+B}(x_1+x_2+x_3+x_4)=\mu_{A+B}\{(x_1+x_2)+(x_3+x_4)\}$$

$$\mu_{A+B}\{(x_1+x_2)+(x_3+x_4)\}\geq \min\{\mu_A(x_1+x_2),\mu_B(x_3+x_4)\}$$

$$\geq \min\{\min(\mu_A(x_1),\mu_A(x_2)),\min(\mu_B(x_3),\mu_B(x_4))\}$$

$$=\min\{\min(\mu_A(x_1),\mu_A(x_2),\mu_B(x_3),\mu_B(x_4))\}$$

$$\therefore \mu_{A+B}(x_1+x_2+x_3+x_4)\geq \min\{(\mu_A(x_1),\mu_B(x_3),\mu_A(x_2),\mu_B(x_4))\}$$

$\therefore A+B+A+B\subset A+B$

Again $k(A+B)=kA+kB$,[$\because kA=A, kB=B$,as A, B are subspaces

$\Rightarrow A+B\subset A+B$

Again kA, is also a fuzzy subspace if A ,is fuzzy subspace.For

(i) $kA+kA=A+A\subset A=kA$

(ii) For any scalar ‘m’ $m(kA)=mA\subset A=kA$,

$\therefore kA$,is a fuzzy subspace.

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