International
Journal
Of Advanced Research in Engineering & Management (IJAREM)

CONTRIBUTION OF RIEMANNIAN GEOMETRY TO QUANTUM THEORY OF LINEAR SIMPLE HARMONIC OSCILLATOR PROBLEM

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Abstract: The linear quantum harmonic oscillator that is well known is built upon Euclidean geometry. In this paper we applied a new geometry called the Riemannian Laplacian to replace the Euclidean Laplacian used in the well-known Schrödinger solution of simple harmonic oscillator. The results obtained here serve as a correction to the well-known eigen energies of the Schrödinger solution.
Keywords—

I Introduction

It is well known how Schrödinger's theory of Quantum mechanics is related to Newton's theory of Classical Mechanics. The mathematical expression of all these theories had been built upon Euclidean geometry [1].
This geometry was introduced by the great and famous Greek Mathematician called Euclid in year 300BC. The Euclidean geometry generated many geometrical quantities such as element of arc vector, element of arc scalar, Euclidean linear velocity vector, Euclidean angular acceleration vector, formulation of Euler's diffusion, Equation formulation of Schrödinger dynamical quantum mechanical wave equation and so on [2].
In the year 1854 Georg Friedrich Bernhard Riemann (1926-1866) introduced the geometry which became synonymous with his name: Riemannian Geometry. This geometry has continued to elude the world precisely because of the lack of knowledge of the Metric tensor(s) of space time in all gravitational fields in nature which contribute the fundamental quantities of Riemannian geometry. In year 2013, a Great Metric Tensor for all gravitational fields nature that is necessary and sufficient for the formation of the oretical physics and mathematical orthogonal curvilinear coordinates in nature was discovered by Professor S.X.K Howusu (in his book titled "Riemannian Revolutions in Physics and Mathematics" [3]. This unique Metric tensor for a all gravitational fields in nature is expressed in term of spherical polar coordinates are:

$$g_{00} = \left(1 + \frac{2}{c^2} f\right) \tag{1}$$

$$g_{11} = \left(1 + \frac{2}{c^2} f\right)^{-1} \tag{2}$$

$$g_{22} = r^2 \tag{3}$$

$$g_{33} = r^2 \sin^2 \theta \tag{4}$$

and

$$g_{uv} = 0\,;\ \ otherwise \tag{5}$$

In recent paper [4], this Metric tensor was used to formulate a Riemannian Laplacian in Cartesian coordinate for all gravitation field in nature. The results obtained for the Riemannian Laplacian in one-dimension, say x-direction is given as:

$$\nabla^2_{\mathrm{R_x}} = -\left(1 + \frac{2}{c^2} f\right)^{-1} \frac{\partial^2}{(\partial x^{\circ})^2} + \left(1 + \frac{2}{c^2} f\right) \frac{\partial^2}{\partial x^2} \tag{6}$$

where f is the gravitational potential field if $f = 0$, then:

$$\nabla^2_{\mathrm{R_x}} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \tag{7}$$

In Einstein Coordinates:
or

$$\nabla^2_{\mathrm{R_x}} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \tag{8}$$

In Minkowski Coordiantes
In this paper, this new Riemannian Laplacian is used in the Schrödinger equation of simple harmonic oscillator problem to determine the eigenenergies and eigen function.

II Mathematical Analysis

The great interest in the generation of the Schrödinger quantum Mechanical energy wave equation for the linear simple harmonic oscillator is well known for non-zero mass, m_0, in one dimensional Hooke's field having natural frequency ω_0, along the $x - axis$. it classical potential energy V is given by [5].

$$V(x) = \frac{1}{2} m_0 {\omega_0}^2 x^2 \qquad (9)$$

The time-dependent Schrödinger's equation in one dimension is given as:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m_0}\nabla_x^2 + \frac{1}{2} m_0 {\omega_0}^2 x^2\right]\psi(x,t) \qquad (10)$$

where

$\nabla_x^2 = \frac{\partial^2}{\partial x^2}$ is called the Euclidean Laplacian built upon the Euclidean geometry. By apply Riemannian Laplacian in (10), we have:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m_0}\nabla_{R_x}^2 + \frac{1}{2} m_0 {\omega_0}^2 x^2\right]\psi(x,t) \qquad (11)$$

where

$\nabla_{R_x}^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$ (in Minkowski Coordinates) is called the Riemannian Laplacian built upon the Riemannian geometry.

Hence, (ii) can be expressed as:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m_0}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right) + \frac{1}{2} m_0 {\omega_0}^2 x^2\right]\psi(x,t) \qquad (12)$$

where:

$\psi(x,t)$ is the quantum mechanical wave function.

Now, we seek method of separation of variable as;

$$\psi(x,t) = X(x)\exp\left[-\frac{iEt}{\hbar}\right] \qquad (13)$$

where E is the quantum mechanical energy and $X(x)$ is the quantum mechanical energy wave function. It follows form (13), (12) becomes:

$$0 = -\frac{\hbar^2}{2m_0}X''_{(x)} + \left[\frac{E^2}{2m_0 c^2} - E + \frac{1}{2} m_0 {\omega_0}^2 x^2\right]X(x) \qquad (14)$$

Let ξ be a new independent variable defined by:

$$\xi = \left(\frac{m_0\omega_0}{\hbar}\right)^{\frac{1}{2}} x \qquad (15)$$

Then, the quantum mechanical energy wave equation for the linear simple harmonic oscillator (14) transform as:

$$0 = -X''(\xi) + \left[\frac{2E}{\hbar\omega_0} - \frac{E^2}{2\hbar m_0 \omega_0 c^2} - \xi^2\right]X(\xi) \qquad (16)$$

or

$$0 = X''(\xi) + [\lambda - \xi^2]X(\xi) \qquad (17)$$

where

$$\lambda = \frac{2E}{\hbar\omega_0} - \frac{E^2}{2\hbar m_0 \omega_0 c^2} \qquad (18)$$

Next, we seek the solution of (17) in the form:

$$X(\xi) = \exp\left[-\frac{1}{2}\xi^2\right]F(\xi) \qquad (19)$$

Then the function $F(\xi)$ satisfies the equation:

$$0 = -F''(\xi) - 2\xi F'(\xi) + (\lambda - 1)F(\xi) \qquad (20)$$

This is the Hermit eigen equation with eigenvalues:

$(\lambda - 1) = 2n\,; \quad n = 0, 1, 2\ \ldots$

or

$$\lambda = 2n + 1\,; \quad n = 0, 1, 2\ \ldots \tag{21}$$

It follows from (18), (21) becomes:

$$E^2 - 2m_0c^2E + \hbar m_0\omega_0c^2(2n+1) = 0 \tag{22}$$

We seek solution of (22) by using quadratic formula given as:

$$E = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{23}$$

where

$$a = 1 \tag{24}$$

$$b = -2m_0c^2 \tag{25}$$

and

$$c = \hbar m_0\omega_0c^2(2n+1) \tag{26}$$

It follows from (24) – (26) in (23); we have

$$E_n = m_0c^2\left[1 \pm \left(1 - \frac{(2n+1)\hbar\omega_0}{m_0c^2}\right)^{\frac{1}{2}}\right] \tag{27}$$

or

$$E_n = m_0c^2\left[1 + \left(1 - \frac{(2n+1)\hbar\omega_0}{m_0c^2}\right)^{\frac{1}{2}}\right] \tag{28}$$

Or

$$E_n = m_0c^2\left[1 - \left(1 - \frac{(2n+1)\hbar\omega_0}{m_0c^2}\right)^{\frac{1}{2}}\right] \tag{29}$$

Here, we choose (29) as the solution. Then it follows by binomial expansion, (29) becomes:

$$E_n = \frac{(2n+1)}{2}\hbar\omega - \frac{(2n+1)^2}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \ \ldots \quad n = 0, 1, 2, \ldots \tag{30}$$

where

E_n is the generalized eigen energies of the simple harmonic oscillator

III Results and Discussion

The general eigen energies is determined by using binomial expansion in (29), we have:

$$E_n = \frac{(2n+1)}{2}\hbar\omega_0 - \frac{(2n+1)^2}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \ \ldots \quad n = 0, 1, 2, \ldots \tag{31}$$

Hence, the ground energy level is obtained when $n = 0$; (31) becomes:

$$E_0 = \frac{1}{2}\hbar\omega_0 - \frac{1}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \ \ldots \tag{32}$$

And the first energy level is obtained when $n = 1$; (31) becomes:

$$E_1 = \frac{3}{2}\hbar\omega_0 - \frac{9}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \ \ldots \tag{33}$$

Also, the second energy level is obtained when $n = 2$; (31) becomes:

$$E_2 = \frac{5}{2}\hbar\omega_0 - \frac{15}{8}\frac{\hbar^2\omega_0}{m_0c^2} - \ \ldots \tag{34}$$

and so on.

The profound physical result of this paper is the discovery of the corrections of the will known sequences of Schrödinger is quantum Mechanical eigen energies of the linear simple harmonic oscillator which are more significant as the rest mass of the oscillator becomes smaller and smaller especially the subatomic and elementary particles.

Consequently, the resolutions in quantum theory of the linear simple harmonic oscillator in this paper implies corresponding hitherto unknown revolutions in all area of theoretical and experimental physics of oscillations or vibration such as:

- Solid state Physics
- Statistical Physics.
- Thermal Physics.

and most especially, elementary particle Physics.

IV Conclusion

In conclusion, the profound discovery and contribution in this paper can be applied to all Schrödinger's mechanical wave equation for any entity of non-zero rest mass such as:

- Infinite potential well
- Rectangular potential well
- Finite step potential, and so on

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