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The Application of Newton's Iterative Method in Secondary School Mathematics

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Abstract: In the field of mathematics, Newton's iteration method stands as a significant numerical computation technique, possessing profound theoretical significance and extensive practical value. As the Gaokao increasingly demands comprehensive literacy and innovative capabilities from students, the format and content of its mathematics questions continue to evolve. Consequently, examination papers increasingly feature questions centred upon Newton's iteration method. This paper focuses on innovative questions centred around Newton's method, delving into the essence of this iterative technique and providing detailed analysis and critique through representative examples. It aims to assist students in gaining a profound understanding of the problem-solving approach and methodology for such question types, thereby enhancing their problem-solving abilities. Concurrently, it seeks to support teachers in refining their teaching strategies and elevating instructional quality.

Keywords: Newton's iteration method, Derivative， Sequence

1. Introduction

In the field of mathematics, Newton's method stands as a significant numerical computation technique, possessing profound theoretical significance and extensive practical value. In recent years, as the Gaokao increasingly emphasises students' comprehensive literacy and innovative capabilities, examination questions based on Newton's method have progressively appeared in the mathematics papers. Such questions not only assess students' grasp of fundamental concepts such as functions and derivatives, but also place greater emphasis on evaluating their ability to transfer knowledge, demonstrate innovative thinking, and apply integrated knowledge to solve problems.

Newton's method is a numerical calculation technique employed to determine the zero points of a function. Its fundamental principle involves approximating the function curve using the tangent line at a specific point of the function y = $f(x)$. Through iterative refinement, it progressively approximates the function's zero point.

For the equation $f(x) = 0$, let x_0 be an initial approximate root. The function $f(x)$ has a Taylor expansion at x_0: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$. Using a first-order Taylor series expansion to approximate $f(x)$, we have $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. Therefore, the equation$f(x)$=0canbe approximated as $f(x_0) + f'(x_0)(x - x_0) = 0$. When $f'(x_0) \neq 0$, treat its root $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ as an approximate solution to the original equation $f(x)$=0. Repeating the above process yields the iterative formula $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} (k \in N)$. By repeatedly executing this iterative process, x_k will progressively approach the zero of the function $f(x)$ =0.

Furthermore, by drawing tangents to the curve y = f(x) at points (x_k, $f(x_k)$), the x-coordinate of their intersection with the x-axis is precisely the solution $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. Hence, Newton's method is also known as Newton's tangent method.The Newton iteration method offers advantages such as rapid convergence and high precision, yet it possesses certain limitations. Its convergence depends on the properties of the function and the choice of initial value [1].

2. Case Study of Newton's Iterative Method in Higher Education Entrance Examination Questions

2.1 Construct a recursive relationship using Newton's method

Constructing recursive sequences using Newton's iteration method enables examination of sequence convergence, inequality proofs, approximate solutions, and determining general terms and the sum of the first n terms. This approach has frequently appeared in university entrance examinations, where sequences are constructed through Newton's iteration to assess students' flexible application of sequence knowledge [2-4].

Example 1: The sequence $\{x_n\}$ is defined by the following conditions:

$$x_1 = a > 0, x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), n \in N$$

(1) Proof: For $n \geq 2$, there always exists $x_n \geq \sqrt{a}$;

(2) Proof: For $n \geq 2$, there always exists $x_n \geq x_{n+1}$.

Analysis: At first glance, this problem appears unrelated to Newton's iteration method. However, we may transform the recursive formula for the sequence x_n as follows:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) = \frac{x_n^2 + a}{2x_n} = x_n - \frac{x_n^2 - a}{2x_n}$$

Therefore, we may set $f(x) = x^2 - a$. By employing Newton's iteration method, we can derive the iterative formula for the sequence, thereby completing the proof of the problem.

Proof: $f(x) = x^2 - a$.

(1) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right]$

Let $h(x) = \frac{1}{2}\left[x + \frac{a}{x}\right] \cdot h'(x) = \frac{1}{2}\left[1 - \frac{a}{x^2}\right]$. From $h'(x) = 0$, we obtain $x = \sqrt{a}$. So $h(x)$ is monotonically decreasing on $(0, \sqrt{a})$ and monotonically increasing on $(\sqrt{a}, +\infty)$ .Therefore $h(x) \geq h(\sqrt{a}) = \sqrt{a}$, $x_{n+1} \geq \sqrt{a}(n = 1, 2, 3, \cdots)$. which signifies $x_n \geq \sqrt{a}(n = 2, 3, \cdots)$.

(2) $$\frac{x_{n+1}}{x_n} = \frac{\frac{1}{2}\left[x_n + \frac{a}{x_n}\right]}{x_n} = \frac{1}{2}\left[1 + \frac{a}{x_n^2}\right]$$

Since $x_n \geq \sqrt{a}(n = 2, 3, \cdots)$, it follows that

$$\frac{a}{x_n^2} \leq 1(n = 2, 3, \cdots), \frac{x_{n+1}}{x_n} = \frac{1}{2}\left[1 + \frac{a}{x_n^2}\right] \leq \frac{1}{2}[1 + 1] = 1 \cdot$$

Therefore, for $n \geq 2$, $x_n \geq x_{n+1}$ always holds.

Example 2: Given the function $f(x) = x^2 - 4$. Let the tangent line to the curve $y=f(x)$ at the point$(x_n, f(x_n))$ intersect the x-axis at $(x_{n+1},0)$($n \in N$), where x_1 is a positive real number.

(1) Proof: Prove that if $x_1 > 2$, then $x_n > 2$.

(2) If $x_1 = 4$ and let $a_n = \lg \frac{x_n + 2}{x_n - 2}$, prove that the sequence $\{a_n\}$ is a geometric sequence.

Analysis: This question is evidently designed against the backdrop of the geometric interpretation of Newton's method. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}(n \in N)$.

Proof:

$$x_n-2=x_{n-1}-\frac{f(x_{n-1})}{f'(x_{n-1})}-2=x_{n-1}-2-\frac{x_n^2-4}{2x_n}=x_{n-1}-2-\frac{(x_{n-1}-2)(x_{n-1}+2)}{2x_n}=(x_{n-1}-2)\left(1-\frac{x_{n-1}+2}{2x_n}\right)$$

Therefore

$$x_n-2=\frac{(x_{n-1}-2)^2}{2x_{n-1}}$$

Similarly we get

$$x_n+2=\frac{(x_{n-1}+2)^2}{2x_{n-1}}$$

$$1+\frac{4}{x_n-2}=\frac{x_n+2}{x_n-2}=\frac{(x_{n-1}+2)^2}{(x_{n-1}-2)^2}=\frac{(x_{n-2}+2)^{2^2}}{(x_{n-2}-2)^{2^2}}=\cdots=\left(\frac{x_1+2}{x_1-2}\right)^{2^{n-1}} \qquad (*)$$

(1) From equation (*) we obtain $a_n=\lg\frac{x_n+2}{x_n-2}=2^{n-1}\lg\frac{x_1+2}{x_1-2}$, $\frac{a_{n+1}}{a_n}=2$, so sequence $\{a_n\}$ is a geometric sequence.

(2) $\frac{4}{x_n-2}=\left(\frac{x_1+2}{x_1-2}\right)^{2^{n-1}}-1\Rightarrow x_n=\frac{4}{\left(\frac{x_1+2}{x_1-2}\right)^{2^{n-1}}-1}+2$

Since $x_1=4>2$ a, it follows that

$$\frac{x_1+2}{x_1-2}>1,\quad x_n=\frac{4}{\left(\frac{x_1+2}{x_1-2}\right)^{2^{n-1}}-1}+2>2.$$

2.2 Application of the Newton Iterative Method Variation

Example 3: Given the function $f(x) = x^2 - 2x - 3$. The sequence $\{x_n\}$ is defined as follows: $x_1=2$, where x_{n+1} is the x-coordinate of the intersection point of the line passing through points A(4, 5) and B(x_n, $f(x_n)$) with the x-axis.

(1) Proof: $2\le x_n<x_{n+1}<3$;

(2) Proof: Find the general term formula for the sequence $\{x_n\}$.

Analysis: Drawing upon the geometric significance of the single-point secant method, the first question derives the relationship between x_n and x_{n+1} through their iterative formula. This is then combined with mathematical induction to determine the sequence's range and monotonicity. The second question involves solving for the general term based on the relationship obtained from the iterative formula.

Proof: For $f(x) = x^2 - 2x – 3$, the single-point tangent method yields

$$\begin{aligned}x_{n+1}&=x_n-\frac{x_n-x_0}{f(x_n)-f(x_0)}f(x_n)\\&=x_n-\frac{x_n-x_0}{x_n^2-2x_n-3-\left(x_0^2-2x_0-3\right)}\left(x_n^2-2x_n-3\right)\\&=\frac{x_0x_n+3}{x_n+x_0-2}\end{aligned}$$

$$x_{n+1}-x_n=\frac{x_0x_n+3}{x_n+x_0-2}-x_n=\frac{-(x_n^2-2x_n-3)}{x_n+x_0-2}=\frac{-f(x_n)}{x_n+x_0-2}$$

Therefore, the monotonicity of the sequence $\{x_n\}$ depends on $f(x_n)$when x_0 is known. Since $x_0 = 4$ and $x_1 = 2$, we are seeking an approximate solution for 3 in the interval [2, 4]. Hence, we can determine the monotonicity of the sequence $\{x_n\}$.

Conclusions

The Newton's method formula is concise and stands as one of the principal techniques for approximating solutions to equations, maintaining a close connection with the derivatives section of single-variable functions studied at secondary school level. During the iterative process of Newton's method, the resulting sequence can be seamlessly integrated with numerical sequences. Furthermore, the secant method, a generalisation derived from Newton's method, finds application in secondary school mathematics. For topics such as the application of Newton's method, which bridge advanced mathematics with secondary school mathematics, teaching should incorporate the historical development of these concepts within the context of mathematical history. This approach should guide students towards deeper exploration in an accessible manner, thereby enhancing their understanding of mathematical principles and methodologies.

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