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FIXED POINT IN 2-METRIC SPACE FOR NON CONTINUOUS MAP

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Abstract: In this paper we have established fixed point theorem in 2-metric space which generalizes many previous results.
Key Words: Fixed point theorem, 2-metric space, Euclidean space, Rational expression.

1. Introduction:

After the introduction of concept of 2-metric space by Gahler [3] many authors [1] [2] [3] etc. establishes an analogue of Banach. Contraction principle in 2-metric space. First Kannan [6] and later on Jaggi and Das [5] established the fixed point theorem for non-continuous maps in metric space.
In this paper we have extended this idea to 2-metric space in more general form by increasing the number of terms in R.H.S.

2. Preliminaries:

Now we give some basic definitions and well known results that are needed in the sequel.

Definition (2.1): [3] let X be an non empty set and d:XxXxX→R_+.
If for all x, y, z and u in X, we have
(d_1) d(x,y,z) = 0 if at least two of x, y, z are equal.
(d_2) for all x≠y there exist a point z in X st. d(x,y,z)≠0.
(d_3) d(x,y,z) = d(x,z,y) = d(y,z,x) = ………………… and so on.
(d_4) d(x,y,z) ≤ d(x,y,u) + d(x,u,z) + d(u,y,z)
Then d is called a 2-metric on X and the pair (X,d) is called a 2-metric space.

Definition (2.2): A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X,d) is said to be a Cauchy sequence if $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} d(x_n,x_m,a)=0$ for all $a \in X$.

Definition (2.3): A sequence $\{x_n\}_{n\in N}$ in a 2-metric space (X,d) is said to be a convergent if $\lim\limits_{n\to\infty} d(x_n,x,a)=0$ for all $a \in X$. The point x is called the limit of the sequence.

Definition (2.4): A 2-metric space (X,d) is said to be complete if every Cauchy sequence is X in convergent.

3. Main Result:

Theorem 3.1: Let (X,d) be a complete 2-metric space and Let T:X→X satisfying.

$$d(Tx,Ty,a) \leq \alpha\frac{d(x,Tx,a)[1+d(y,Ty,a)]}{1+d(Tx,Ty,a)}+\beta\frac{d(x,Tx,a)[1+d(x,Tx,a)]}{1+d(x,y,a)} + \gamma\frac{d(y,Ty,a)[1+d(x,Tx,a)]}{1+d(x,y,a)}+\delta d(x,y,a)............(3.1.1)$$

for all $x,y,a \in X$ and $d(Tx,Ty,a)\neq 0$, $d(x,y,a)\neq 0$. Also $\alpha,\beta,\gamma,\delta \geq 0$ and $\alpha+\beta+\gamma+\delta<1$. Then T has a unique fixed point.

Proof: Let for any arbitrary point $x_0 \in X, \{x_n\}$ be a Cauchy sequence defined as:

x_0, $x_1 = Tx_0$: $x_2 = Tx_1$, ..., $x_n = Tx_{n-1}$, $x_{n+1} = Tx_n$, ...
Then,

$d(x_1,x_2,a)$ = $d(Tx_0,Tx_1,a)$

$$\leq \frac{\alpha d(x_0,Tx_0,a)[1+d(x_1,Tx_1,a)]}{[1+d(Tx_0,Tx_1,a)]} + \frac{\beta d(x_0,Tx_0,a)[1+d(x_0,Tx_0,a)]}{[1+d(X_0,x_1,a)]}$$

$$+\frac{\gamma d(x_1,Tx_1,a)[1+d(x_0,Tx_0,a)]}{[1+d(x_0,x_1,a)]} + \delta d(x_0,x,a)$$

$$= \frac{\alpha d(x_0,x_1,a)[1+d(x_1,x_2,a)]}{[1+d(x_1,x_2,a)]} +$$

$$+\frac{\beta d(x_0,x_1,a)[1+d(x_0,x_1,a)]}{[1+d(x_0,x_1,a)]} + \frac{\gamma d(x_1,x_2,a)[1+d(x_0,x_1,a)]}{[1+d(x_0,x_1,a)]}$$

$$+\delta d(x_0,x_1,a).$$

$$= \alpha d(x_0,x,a) + \beta d(x_0,x_1,a) + \gamma d(x_1,x_2,a) + \delta d(x_0,x_1,a)$$

$$(1-\gamma)d(x_1,x_2,a) \leq (\alpha+\beta+\delta)d(x_0,x_1,a)$$

or, $d(x_1,x_2,a) \leq \left(\frac{\alpha+\beta+\delta}{1-\gamma}\right)d(x_0,x_1,a)$

Again, $d(x_2,x_3,a) = d(Tx_1,Tx_2,a)$

$$\leq \alpha \frac{d(x_1,T x_1,a)[1+d(x_2,T x_2,a)]}{[1+d(T x_1,T x_2,a)]}$$

$$+\beta \frac{d(x_1,T x_1,a)[1+d(x_1,T x_1,a)]}{[1+d(x_1,x_2,a)]}$$

$$+\gamma \frac{d(x_2,T x_2,a)[1+d(x_1,T x_1,a)]}{[1+d(x_1,x_2,a)]} + \delta d(x_1,x_2,a)$$

$$= \alpha \frac{d(x_1,x_2,a)[1+d(x_2,x_3,a)]}{[1+d(x_2,x_3,a)]} + \beta \frac{d(x_1,x_2,a)[1+d(x_1,x_2,a)]}{[1+d(x_1,x_2,a)]}$$

$$\gamma \frac{d(x_2,x_3,a)[1+d(x_1,x_2,a)]}{[1+d(x_1,x_2,a)]} + \delta d(x_1,x_2,a)$$

or $(1-\gamma)d(x_2,x_3,a) \leq (\alpha+\beta+\delta)d(x_1,x_2,a)$

or $d(x_2,x_3,a) \leq \frac{(\alpha+\beta+\delta)}{1-\gamma} d(x_1,x_2,a)$

$$\leq \left(\frac{\alpha+\beta+\delta}{1-r}\right)^2 d(x_1,x_2,a)$$

Similarly we have,

$$d(x_n,x_{n+1},a) \leq \left(\frac{\alpha+\beta+\delta}{1-\gamma}\right)^n d(x_0,x_1,a)$$

$$= K^2 d(x_0,x_1,a), \text{where } K = \frac{\alpha+\beta+\delta}{1-\gamma} \qquad - \qquad (3.2.2)$$

Now we claim that $d(x_0,x_1,x_n)=0$ for $n=0,1,2,\ldots\ldots\ldots$ - (3.2.3)

We observe that this is true for $n=o$ and $n=1$. Suppose that it is true for $2 \leq n \leq m$. Using triangle in equality we have

$$0 \le d(x_0, x_1, x_{m+1}) \le d(x_0, x_1, x_m) + d(x_0, x_m, x_{m+1}) + d(x_m, x_1, x_{m+1})$$
$$\le \kappa^m [d(x_0, x_0, x_1) + d(x_0, x_1, x_1)]$$
$$= 0$$

Since $[d(x_n, x_{n+1}, x_{n+p}) \le k^n d(x_0, x_1, x_{n+p})$ it is follows from (2.2.3) that $d(x_n,x_{n+1},x_{n+p})=0$ for all non-negative integers m and n.

We now show that $\{x_n\} n \in N$ is a Cauchy sequence. For arbitrary $a \in X$, we have,

$$d(x_n, x_{n+p}, a) \le d(x_n, x_{n+1}, x_{n+p}) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, x_{n+p})$$
$$+ d(x_{n+1}, x_{n+2}, a) ....$$
$$+ d(x_{n+p-2}, x_{n+p-1}, x_{n+p}) + d(x_{n+p-1}, x_{n+p}, a)$$
$$= [k^n + k^{n+1} + ... + k^{n+p-1}]\, d(x_0, x_1, a)$$
$$= \left(\frac{k^n}{1-k}\right) d(x_0, x_1, a)$$
$$\to 0 \text{ as } n \to \infty \text{ since } K<1.$$

This shows that $\{x_n\} n \in N$ is a Cauchy sequence. Since X is complete. Therefore there exists a point u in X s.t.

$\lim_{n\to\infty} x_n = u$ Now we shall prove that $Tu = u$

$$d(u, Tu, a) \le d(u, Tu, x_n) + d(u, x_n, a) + d(x_n, Tu, a)$$
$$= d(u, Tu, x_n) + d(u, x_n, a) + d(T x_{n-1}, Tu, a)$$
$$\le d(u, Tu, x_n) + d(u, x_{2n}, a) + \alpha \frac{d(x_{n-1}, T x_{n-1}, a)[1 + d(u, Tu, a)]}{1 + d(T x_{n-1}, Tu, a)}$$
$$+ \beta \frac{d(x_{n-1}, T x_{n-1}, a)[1 + d(x_{n-1}, T x_{n-1}, a)]}{1 + d(x_{n-1}, u, a)}$$
$$+ \gamma \frac{d(u, Tu, a)[1 + d(x_{n-1}, T x_{n-1}, a)]}{1 + d(x_{n-1}, u, a)} + \delta d(x_{n-1}, u, a)$$
$$= d(u, Tu, x_n) + d(u, x_n, a) +$$
$$\frac{\alpha d(x_{n-1}, x_n, a)[1 + d(u, Tu, a)]}{1 + d(x_n, Tu, a)} + \frac{\beta d(x_{n-1}, x_n, a)[1 + d(x_{n-1}, x_n, a)]}{1 + d(x_{n-1}, Tu, a)}$$
$$\frac{\gamma d(u, Tu, a)[1 + d(x_{n-1}, x_n, a)]}{1 + d(x_{n-1}, u, a)} + \delta d(x_{n-1}, u, a)$$

When $n \to \infty$, we have

$$d(u, Tu, a) \le \gamma d(u, Tu, a)$$

or $(1-\gamma) d(u, Tu, a) \le 0$, which implies that of $d(u, Tu, a) = 0$ i.e. $Tu = u$.

Now we shall show that u is a unique fixed point of T. If not, possible let $v \ne u$ be another fixed point such that $Tv = v$

$$d(u, v, a) = d(Tu, Tv, a)$$

$$\leq \frac{d(u,Tu,a)[1+d(v,Tv,a)]}{1+d(Tu,Tv,a)}+\frac{\beta d(u,Tu,a)[1+d(u,Tu,a)]}{1+d(u,v,a)}$$
$$+\frac{\gamma d(v,Tv,a)[1+d(u,Tu,a)]}{1+d(u,v,a)}+\delta d(u,v,a)$$

Or, $d(u,v,a)\leq \delta d(u,v,a)$

Or, $(1-\delta)d(u,v,a)\leq 0$, which implies that $d(u,v,a) = 0$ i.e. $u=v$

Thus u is a unique fixed point of T.

Corollary (3.2): Let T^p be sequence of mapping on a complete 2-metric space X into itself satisfying.

$$d(T^p x,T^p y,a)\leq \alpha\frac{d(x,T^p x,a)[1+d(y,T^p y,a)]}{1+d(T^p x,T^p y,a)}+\beta\frac{d(x,T^p x,a)[1+d(x,T^p x,a)]}{1+d(x,y,a)}$$
$$+\gamma\frac{d(y,T^p y,a)[1+d(x,T^p x,a)]}{1+d(x,y,a)}+\delta d(x,y,a)$$

for all x, y, a in X. where $d(Tx,Ty,a)\neq 0$ and $d(x,y,a)\neq 0$. Also $\alpha,\beta,\gamma,\delta\geq 0$ where $\alpha,\beta,\gamma,\delta<1$ and p is a positive integer.

Example (3.3): Let (X,d) be a 2-metric space defined as follow:

$$d(x,y,z)=min\{|x-y|,|y-z|,|z-x|\}$$

Now let X=[0,1] with the 2-metric defined as above.

Let $T:X\rightarrow X$ defined as:

$$Tx=\begin{cases} x/4 : x\in\left[0,1/2\right) \\ x/5 : x\in\left[1/2,1\right) \end{cases}$$

Now taking $\alpha=1/2,\beta=1/3,\gamma=1/5,\delta=1/7$ and $a=1, x=1/3, y=1/2$

Then we have

$d(Tx,Ty,1)=d(1/12,1/6,1)=$ $min\left\{\left|\frac{1}{12}-\frac{1}{6}\right|\left|\frac{1}{6}-1\right|\left|1-\frac{1}{12}\right|\right\}=min\left\{\frac{1}{12},\frac{5}{6},\frac{11}{12}\right\}=\frac{1}{12}\approx 0.083\bar{3}$

Now, $d(x,Tx,1)=d\left(1/3,1/12,1\right)=min\left\{\left|\frac{1}{3}-\frac{1}{12}\right|,\left|\frac{1}{12}-1\right|,\left|1-\frac{1}{3}\right|\right\}=min\left\{\frac{1}{4},\frac{11}{12},\frac{2}{3}\right\}=\frac{1}{4}$

$$d(y,Ty,1)=d\left(1/2,1/10,1\right)=min\left\{\left|\frac{1}{2}-\frac{1}{10}\right|,\left|\frac{1}{10}-1\right|,\left|1-\frac{1}{2}\right|\right\}=min\left\{\frac{2}{5},\frac{9}{10},\frac{1}{2}\right\}=\frac{2}{5}$$

$$d(x,y,1)=d\left(1/3,1/2,1\right)=min\left\{\left|\frac{1}{3}-\frac{1}{2}\right|\left|\frac{1}{2}-1\right|\left|1-\frac{1}{3}\right|\right\}=min\left\{\frac{1}{6},\frac{1}{2},\frac{2}{3}\right\}=\frac{1}{6}$$

Now,

$$\left\{\alpha\frac{d(x,Tx,1)[1+d(x,Tx,1)]}{1+d(Tx,Ty,1)}+\beta\frac{d(x,Tx,1)[1+d(x,Tx,1)]}{1+d(x,y,1)}+\gamma\frac{(y,Ty,1)[1+d(x,Tx,1)]}{1+d(x,y,1)}+\delta d(x,y,1)\right\}$$

$$=\frac{1}{2}\left\{\left[\frac{\frac{1}{4}\left[1+\frac{2}{5}\right]}{1+\frac{1}{12}}\right]+\frac{1}{3}\left[\frac{\frac{1}{4}\left[1+\frac{1}{4}\right]}{1+\frac{1}{6}}\right]+\frac{1}{5}\left[\frac{\frac{2}{5}\left[1+\frac{1}{4}\right]}{1+\frac{1}{6}}\right]+\frac{1}{7}\times\frac{1}{6}\right\}=\left\{\frac{21}{130},\frac{5}{56},\frac{3}{35},\frac{1}{42}\right\}\approx 0.360348$$

Again if we put a=0, then also we have,

$d(Tx,Ty,0)=d\left(\frac{1}{12},\frac{1}{6},0\right)=min\left(\left|\frac{1}{12}-\frac{1}{6}\right|,\left|\frac{1}{6}-0\right|,\left|0-\frac{1}{12}\right|\right)=$ $min\left\{\frac{1}{12},\frac{5}{6},\frac{1}{12}\right\}=\frac{1}{12}=0.083\bar{3}$

Now,

$d(x,Tx,0)=d\left(\frac{1}{3},\frac{1}{12},0\right)=min\left(\left|\frac{1}{3}-\frac{1}{12}\right|,\left|\frac{1}{12}-0\right|,\left|0-\frac{1}{3}\right|\right)=min\left\{\frac{1}{4},\frac{1}{12},\frac{1}{3}\right\}=\frac{1}{12}$

$$d(y,Ty,0) = d\left(\frac{1}{12},\frac{1}{6},0\right) = min\left(\left|\frac{1}{12}-\frac{1}{6}\right|,\left|\frac{1}{6}-0\right|,\left|0-\frac{1}{12}\right|\right) = min \left\{\frac{1}{12},\frac{1}{6},\frac{1}{12}\right\} = \frac{1}{12}$$

$$d(x,y,0) = d\left(\frac{1}{3},\frac{1}{2},0\right) = min\left\{\left|\frac{1}{3}-\frac{1}{2}\right|\left|\frac{1}{2}-0\right|\left|0-\frac{1}{2}\right|\right\} = min\left\{\frac{1}{6},\frac{1}{2},\frac{1}{3}\right\} = \frac{1}{6}$$

$$R.H.S. = \left\{\frac{1}{2}\left[\frac{\frac{1}{12}\left[1+\frac{1}{12}\right]}{1+\frac{1}{12}}\right]+\frac{1}{3}\left[\frac{\frac{1}{12}\left[1+\frac{1}{12}\right]}{1+\frac{1}{6}}\right]+\frac{1}{5}\left[\frac{\frac{1}{12}\left[1+\frac{1}{12}\right]}{1+\frac{1}{6}}\right]+\frac{1}{7}\times\frac{1}{6}\right\}$$
$$\approx 0.102467$$

Thus we see that the condition of our theorem is satisfied with 0 (zero) as the only fixed point while T is not contraction (being discontinuous).

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